On the statistical leak of the GGH13 multilinear map and its variants

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Introduction

In this talk:

- Focus on the GGH13 multilinear map

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- Focus on the GGH13 multilinear map
- Classical attacks: zeroizing attacks
  $\Rightarrow$ main application of GGH today: obfuscators
- Contribution: analyze averaging attacks
  - In some case, we have a complete attack against GGH.
  - In some other cases, we get some leaked information.

Table of Contents

1. The GGH13 multilinear map
2. Zeroizing attacks and consequences
3. Averaging attacks

History of multilinear maps (until February 2015)

- 2000 Joux introduces bilinear maps (pairings) for cryptographic uses.
- 2003 Boneh and Silverberg introduce the concept of multilinear maps.
- $\geq$ 2003 Many applications.
- 2013 Garg, Gentry and Halevi publish the first candidate multilinear map (GGH13 map).
- 2013 Garg et al. publish the first candidate obfuscator, using the GGH13 map.
- 2013 Coron, Lepoint and Tibouchi propose another candidate multilinear map, relying on integers (CLT map).
- 2015 Gentry, Gorbunov and Halevi propose a graph-induced multilinear map (GGH15 map).

Cryptographic multilinear maps

Definition: κ-multilinear map

Different levels of encodings, from 0 to κ.
Denote by $C(a, i)$ a level-i encoding of the message a.
Level-0 encoding: a plaintext (message not encoded).

Addition: $\mathrm{Add}(C(a_1, i), C(a_2, i)) = C(a_1 + a_2, i)$.

Multiplication: $\mathrm{Mult}(C(a_1, i), C(a_2, j)) = C(a_1 \cdot a_2, i + j)$.

Zero-test: $\text{Zero-test}(C(a, \kappa)) = \text{True iff } a = 0$.

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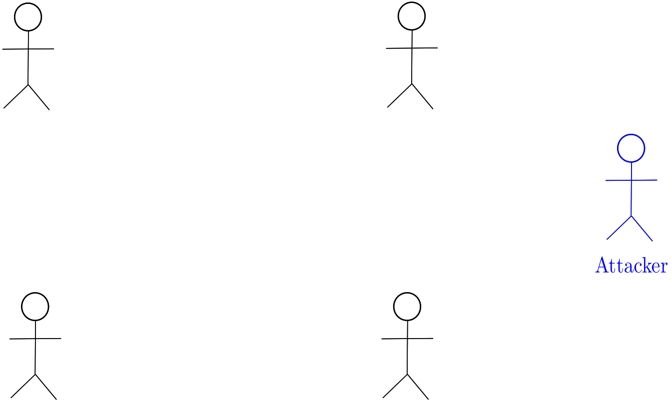
Security: What should be hard for a cryptographic multilinear map?

Application to multipartite key-exchange

Objective: $\kappa + 1$ users want to agree on a shared secret s.
Let D be a distribution over the message space.



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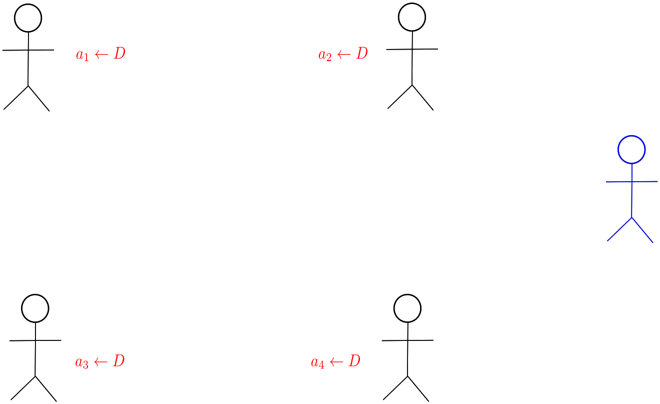

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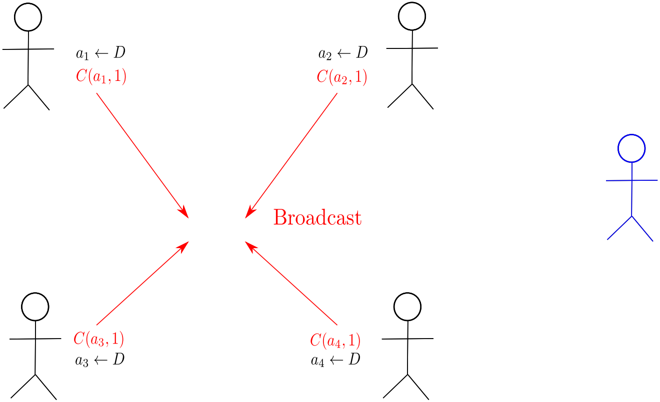

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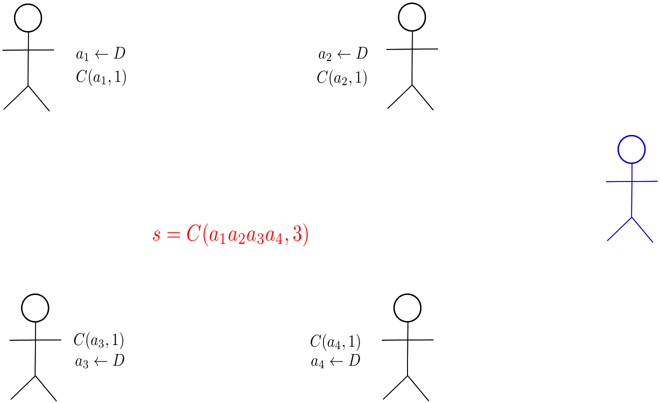

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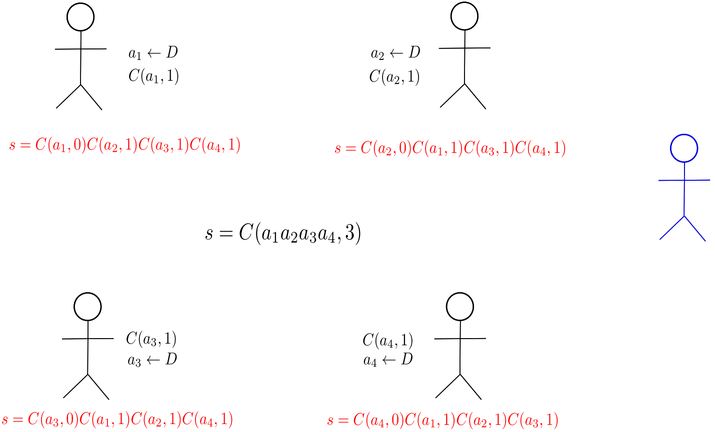

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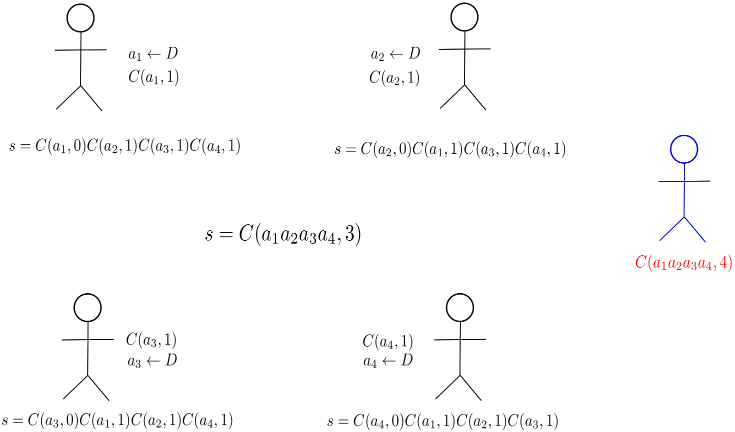

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- Define $R = \mathbb{Z}[X]/(X^n+1)$ with $n = 2^k$.
- Sample g a "small" element in R.
  $\Rightarrow$ the plaintext space is $\mathcal{P} = R/\langle g\rangle$.
- Sample q a "large" integer.
  $\Rightarrow$ the encoding space is $R_q = R/(qR) = \mathbb{Z}_q[X]/(X^n+1)$.

Notation

We write $[r]_q$ or $[r]$ the elements in R_q, and r (without $[\cdot]$) the elements in R.

The GGH13 multilinear map: encodings

- Sample z uniformly in R_q.
- **Encoding:** An encoding of a at level i is

$$u = [(a + rg)z^{-i}]_q$$

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Addition and multiplication

Addition:

$$[(a_1 + r_1 g)z^{-i}]_q + [(a_2 + r_2 g)z^{-i}]_q = [(a_1 + a_2 + r' g)z^{-i}]_q.$$

Multiplication:

$$[(a_1 + r_1 g)z^{-i}]_q \cdot [(a_2 + r_2 g)z^{-j}]_q = [(a_1 \cdot a_2 + r' g)z^{-(i+j)}]_q.$$

The GGH13 multilinear map: zero-test

- Sample h in R of the order of $q^{1/2}$.
- Define

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Zero-test

To test if $u = [cz^{-\kappa}]$ is an encoding of zero (i.e. $c = 0 \mod g$), compute
$$[u \cdot p_{zt}]_q = [chg^{-1}]_q.$$
This is small iff c is a small multiple of g.

The GGH13 multilinear map: other public parameters

Question

How to compute an encoding of a at level 1 when we only have the public parameters R, q and p_{zt}?

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Solution. We add to the public parameters

- y an encoding of 1 at level 1
- x an encoding of 0 at level 1.

To compute $C(a, 1)$:

Sample r in R and output $u = [ay + rx]_q$.

Conclusion on the GGH13 map

- We have a mathematical object, that satisfies some properties (addition, multiplication, zero-test).
- What about its security ?

Table of contents: 2 - Zeroizing attacks and consequences

1 The GGH13 multilinear map

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Zeroizing attacks

Idea

When $u = [cz^{-\kappa}]_q$ with $c = bg$ a small multiple of g, we have

$$[u \cdot p_{zt}]_q = [chg^{-1}]_q = bh$$

because bh is smaller than q so $[bh]_q = bh \in R$.

Example of attack (from GGH13)

Compute

$$[x^2 y^{\kappa-2} p_{zt}]_q = [g^2 \cdot r \cdot g^{-1}]_q = g \cdot r$$

$\Rightarrow$ recover multiples of g, and then $\langle g \rangle$.

Hu and Jia's attack

Hu and Jia, 2015[1]

An attacker can recover the shared secret s in the multipartite key exchange protocol, when using the GGH13 multilinear map.

For this attack, we need x, the level 1 encoding of zero.

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Question

Maybe the GGH13 map is still safe if we do not have low level encodings of zero?

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Not all obfuscators are broken yet

Good news for obfuscators

We do not need the public parameters x and y in the GGH13 map when used for obfuscators.

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Yes but...

Still, many obfuscators using the GGH13 map were proven insecure using zeroizing techniques.

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Another approach: averaging

Idea

Instead of looking at the arithmetic properties of R, we use statistical properties.

This kind of attacks was already mentioned in the original article of GGH13.

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Property: If D is a distribution over R and $x_1, \cdots, x_\ell$ are independent elements sampled from D, then

$$\frac{1}{\ell}\sum_{i=1}^{\ell} x_i \underset{\ell \to +\infty}{\longrightarrow} \mathbb{E}(x_1).$$

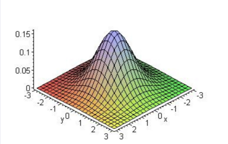
With ℓ samples, we expect to get $\log(\ell)$ bits of precision for $\mathbb{E}(x_1)$.

Notations and definitions (1)

Definitions

A distribution is said **centered** if its mean is zero.
A distribution is said **isotropic** if no direction is privileged.

Example



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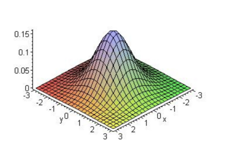

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Gaussian distribution

We denote by D_σ the (discrete) Gaussian distribution centered in 0 and of variance σ^2.

Remark. D_σ is a centered isotropic distribution (if σ is large enough).

Definitions and properties (2)

Definitions / Notation

- For $r \in R$, we denote $A(r) = r\bar{r}$ the **auto-correlation** of r, where $\bar{r}$ is the complex conjugate of r when seen in $\mathbb{C}$.
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- The **variance** of a centered variable r is $\mathrm{Var}(r) := \mathbb{E}(r\bar{r})$.

Proposition: If r is sampled in R according to a centered isotropic distribution, then

$$\mathbb{E}(r) = 0$$
$$\mathrm{Var}(r) = \mu \in \mathbb{R}$$

Back to the attack: what do we know?

Reminder: We do not want to publicly give x and y anymore. So what is public?

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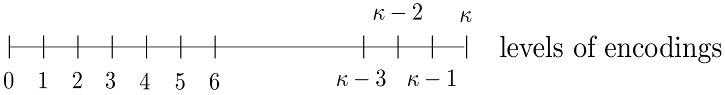
- we are given R, q and p_{zt} as before.
- we are given $u_i = [c_i z^{-i}]$ for $1 \leq i < \kappa$ and $c_i \leftarrow D_\sigma$.
- such that $u_i u_{\kappa-i}$ is an encoding of 0 at level κ.



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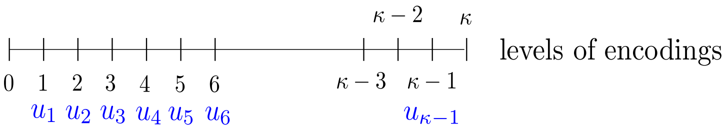

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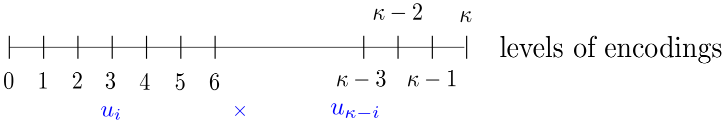

Idea of the attack

Recall our model

- we are given $u_i = [c_i z^{-i}]$ for $1 \leq i \leq \kappa - 1$ and $c_i \leftarrow D_\sigma$.
- such that $u_i u_{\kappa-i}$ is an encoding of 0 at level κ.

Observation:

$$\begin{aligned}[u_i u_{\kappa-i} \cdot p_{zt}] &= [c_i c_{\kappa-i} \cdot h/g] \\ &= c_i c_{\kappa-i} \cdot h/g \\ &= c_i^* \cdot h/g\end{aligned}$$

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- $\mathrm{Var}(c_i^*) = \mathbb{E}(A(c_i^*)) = \mu \in \mathbb{R}$ is some scalar $\Rightarrow$ we obtain

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$$\frac{1}{\kappa}\sum_{i=1}^{\kappa} A(c_i^* \cdot h/g) \underset{\kappa \to +\infty}{\to} \mu A(h/g).$$

We get an approximation of $A(h/g)$ with $\log(\kappa)$ bits of precision.

GGH13 counter-measure

GGH13's authors noticed that their scheme was subject to averaging attacks $\Rightarrow$ they proposed a countermeasure.

Definition

Let z_i be the representative of $[z^i]$ in R with coefficients in $[-q/2, q/2]$.

Idea: choose c_i such that c_i/z_i is isotropic.

Counter-measure

- Sample $\widetilde{c_i} \leftarrow D_\sigma$.
- Define $c_i = \widetilde{c_i} \cdot z_i$.
- And $u_i = [c_i z^{-i}]$ as before.

Adapting the attack to the counter-measure

Recall

- $c_i = \widetilde{c_i} \cdot z_i$.
- $u_i = [c_i z^{-i}]$.
- $u_i u_{\kappa-i}$ is an encoding of 0 at level κ.

Observation:

$$\begin{aligned} [u_i u_{\kappa-i} \cdot p_{zt}] &= \widetilde{c_i}\widetilde{c_{\kappa-i}} \cdot z_i z_{\kappa-i} \cdot h/g \\ &= c_i^* \cdot z_i z_{\kappa-i} \cdot h/g \end{aligned}$$

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But: the z_i are isotropic and independent.

Averaging: we get an approx of $\mu A(h/g)$, for some constant μ.

Conclude the attack

Lemma

If we have

- an approximation of $A(h/g)$ with $\log \ell$ bits of precision,
- a guarantee that for any encoding $[cz^{-i}]$, the coefficients of c are less than $\ell/2$.

Then, we can recover $A(h/g)$ exactly and attack the GGH13 map.

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Do we get enough samples for recovering $A(h/g)$ exactly?

- Without the counter-measure $\Rightarrow$ yes.
- With the counter-measure $\Rightarrow$ no, but this is because of constraints in the sampling procedure.

Conclusion

In the case where q is polynomial:

- complete attack without the counter-measure (if κ is large enough).
- leaked information with the counter-measure.
- other variants (adapted from [DGG+16][2]): leaked information but no complete attack.

[2] Döttling, N. et al. "Obfuscation from Low Noise Multilinear Maps".

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Thank you for your attention.

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